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Lie algebra and the quantisation of momenta and the Hamiltonian

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Abstract. A common assumption that quantisation is simply a representation of the Lie algebra of classical observables by a Lie algebra of self-adjoint operators in Hilbert space is shown to be generally invalid.

1. Introduction

Let the classical configuration space M be an N-dimensional Riemannian manifold with metric g^{mn}. The phase space is then the contangent bundle T^*M. If x^i is a coordinate system in M, T^*M may be coordinated by (x^i, p_i), p_i being generalised momenta. We shall confine ourselves to classical momentum observables of the form $P = \eta^i(x)p_i$. The set $\{P\}$ of all such P is a Lie algebra with respect to the Poisson bracket operation. The question then arises as to whether there exists a representation of $\{P\}$ by a Lie algebra $\{\mathrm{P}\}$ of self-adjoint operators P in an appropriate Hilbert space $\mathscr{H}$. An affirmative answer commonly assumed would mean that the quantisation of $\{P\}$ is simply a representation of $\{P\}$ by $\{\mathrm{P}\}$ in $\mathscr{H}$. (Hermann 1966, 1970; Bloore and Ghobrial 1975). However we shall show that the question posed has the negative answer. Consequently the above idea of quantisation is not generally valid.

2. Lie algebra and the quantisation of momenta

Our starting point is Mackey's method of quantisation (Mackey 1963). In a recent paper (Wan and Viazminsky 1977), hereafter called I, Mackey's method is explained and applied to the quantisation in spaces of constant curvature. The scheme begins with the well-known link between the function $P = \eta^i p_i$ on T^*M and the vector field $V = \eta^i \partial/\partial x^i$ on M, a result established in the theory of differentiable manifolds (Loomis and Sternberg 1968). This link enables us to classify P according to whether its associated vector field is complete. Let us write $\{P\} = \{P\}_c \cup \{P\}_r$, where $\{P\}_c$ is the set of P whose associated vector fields are complete and $\{P\}_r$ denotes the rest of P in $\{P\}$. Every P in $\{P\}_c$ generates a one-parameter group (OPG for short) of transformations of M. Moreover an OPG of M induces an OPG U of unitary transformations of the set

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$L^2(M)$ of square-integrable functions on M. On quantisation P is assumed to go over to the generator P of U. P is automatically a self-adjoint operator in the Hilbert space $L^2(M)$. Explicitly (Mackey 1963, Wan and Viazminsky 1977)

$$\mathrm{P} = -\mathrm{i}\hbar(V + \tfrac{1}{2}\operatorname{div} V)$$

with domain

$$D_{\mathrm{P}} = \{f: f \in C^1(M); f, \mathrm{P}f \in L^2(M)\}.$$

Here $\operatorname{div} V$ is the divergence of V and V is the vector field associated with P, i.e. $P = \eta^i p_i$ and $V = \eta^i \partial/\partial x^i$. The important observation is that P with domain $C_0^\infty(M)$ is essentially self-adjoint if and only if V is a complete vector field (Wan and McFarlane 1979, Mackey 1963, Wan and Viazminsky 1977). Consequently only P in $\{P\}_c$ are quantisable.

We now come to see whether $\{P\}$ constitutes a Lie algebra under Poisson bracket operation. Let $P_1 = \eta^i_{(1)} p_i$ and $P_2 = \eta^i_{(2)} p_i$ be members of $\{P\}_c$. Their Poisson bracket is then

$$P_3 = \{P_1, P_2\} = \eta^i_{(3)} p_i,$$

where

$$\eta^i_{(3)} = \eta^i_{(1),j}\eta^j_{(2)} - \eta^j_{(1)}\eta^i_{(2),j}.$$

A necessary condition for $\{P\}_c$ to form a Lie algebra is that P_3 so obtained must belong to $\{P\}_c$. In other words, the associated vector field $V_3 = \eta^i_{(3)}\, \partial/\partial x^i$ must be complete. On the other hand V_3 is seen to be just the Lie bracket of

$$V_2 = \eta^i_{(2)} \partial/\partial x^i \quad \text{and} \quad V_1 = \eta^i_{(1)} \partial/\partial x^i, \quad \text{i.e.} \quad V_3 = -[V_1, V_2].$$

Therefore the problem reduces to the completeness of the Lie bracket of two given complete vector fields V_1, V_2. The simple answer to this is that the Lie bracket of two complete vector fields is not necessarily complete (Brickell and Clark 1970). Consequently $\{P\}_c$ is not a Lie algebra; neither is the corresponding set of quantised momenta $\{\mathrm{P}\}$ under commutator bracket operation. This shows that the idea of Lie algebra as a general framework for quantisation is not a valid one. $\{P\}$ being a Lie algebra is not particularly significant since not every P in $\{P\}$ is quantisable. The exception is when M is compact, e.g. a space of positive constant curvature like a sphere. Every vector field on a compact manifold is complete (Brickell and Clark 1970). As a result a Lie algebra quantisation scheme may be carried through. But again this is not of fundamental significance since most configuration manifolds of interest including spaces of zero or negative constant curvature are non-compact.

We now examine whether the idea of Lie algebra has any role to play at all. Let the configuration space M be an N-dimensional space of constant curvature K on which there are $\frac{1}{2}N(N+1)$ linearly independent complete Killing vector fields $V_{(\mu)} = \zeta^i_{(\mu)} \partial/\partial x^i$ generating a $\frac{1}{2}N(N+1)$-parameter group of motions of M. The curvature K may be positive, zero or negative. Let

$$\{V\}_k = \{V: V = \lambda^\mu V_{(\mu)}, \lambda^\mu = \text{real}\},$$

$$\{P\}_k = \{P: P = \zeta^i p_i, V = \zeta^i \partial/\partial x^i \in \{V\}_k\},$$

$$\{\mathrm{P}\}_k = \{\mathrm{P}: \mathrm{P} = -\mathrm{i}\hbar\zeta^i \partial/\partial x^i, V = \zeta^i \partial/\partial x^i \in \{V\}_k\}.$$

In other words, $\{V\}_k$ which consists of all linear combinations of $V_{(\mu)}$ is the set of all

Killing vector fields. Under our assumption every V in $\{V\}_k$ is complete. $\{P\}_k$ is the set of classical momenta associated with $\{V\}_k$. $\{\mathrm{P}\}_k$ is the set of corresponding quantised momenta. Observe that the divergence of a Killing vector field vanishes. The Lie bracket of two Killing vector fields is again a Killing vector field. Linear combinations of two Killing vector fields are also Killing vector fields. Hence $\{V\}_k$ constitutes a Lie algebra (Matsushima 1972). Since there is a one-one and onto map between $\{V\}_k$, $\{P\}_k$ and $\{\mathrm{P}\}_k$ we can endow the same Lie algebra structure to $\{P\}_k$ and $\{\mathrm{P}\}_k$.† Therefore we may say that a Lie algebra approach to quantisation is profitable if it is coupled with symmetry considerations. Only those momentum observables associated with certain geometric symmetry of the configuration space form a Lie algebra classically and quantum mechanically.

3. Lie algebra and the quantisation of the Hamiltonian

In I a scheme for the determination of the quantised Hamiltonian H was introduced. The scheme may now be rephrased in the language of Lie algebra. A close examination of I shows that the H is essentially a quadratic Casimir operator C of the Lie algebra $\{\mathrm{P}\}_k$. General expressions may be worked out explicitly. Let us employ the coordinate system x^i in M in terms of which the metric takes the form (Eisenhart 1964)

$$\mathrm{d}S^2 = \mathrm{d}x^i\,\mathrm{d}x^i/(1+\tfrac{1}{4}Kx^ix^i)^2.$$

We shall consider spaces of constant curvature K only and in such spaces the above coordinates exist. A set of $\frac{1}{2}N(N+1)$ independent Killing vector fields is

$$L_i = (\tfrac{1}{4}K[2(x^i)^2 - x^jx^j]+1)\,\partial/\partial x^i + \tfrac{1}{2}Kx^i(x^j - x^i\delta^j_i)\,\partial/\partial x^j,$$
$$L_{ij} = x^i\,\partial/\partial x^j - x^j\partial/\partial x^i \qquad (i<j).$$

In the above expression for L_i summation over j only is implied, i.e. no summation over i is meant. δ^j_i is the Kronecker delta. The vector fields L_i, L_{ij} satisfy the following Lie bracket relations: (Robertson and Noonan 1968)

$$[L_i, L_j] = -KL_{ij},$$
$$[L_i, L_{ij}] = L_j, \qquad \text{(no summation over } i\text{)}$$
$$[L_{il}, L_{lj}] = L_{ij} \qquad \text{(no summation over } l\text{)}$$

These expressions are valid whether or not $i<j$, $i<l$, $l<j$ if we make the identification $L_{ji} = -L_{ij}$ when $i<j$. The rest Lie brackets not derivable from these expressions are all zero. We have now obtained an explicit Lie algebra. The usual procedure may be set in motion to construct a quadratic Casimir operator C_V in terms of the structure constants available (Fonda and Ghirardi 1970). In what follows let the indices i, j, k, l, m, n run from 1 to N and let the indices r, s, t run from $N+1$ to $\frac{1}{2}N(N+1)$. Now relabel L_i, L_{mn}

† Let $V_i (i=1,2,3,\ldots) \in \{V\}_k$ and let their associated classical momenta and quantised momenta be P_i, P_i respectively. Then $[V_1, V_2] = V_3$ implies $\{P_1, P_2\} = -P_3$, $[\mathrm{P}_1, \mathrm{P}_2] = -\mathrm{i}\hbar\mathrm{P}_3$. This leads to the fact that the bracket operations do not appear to preserve the Lie algebra structure of $\{V\}_k$, $\{P\}_k$, $\{\mathrm{P}\}_k$. But this is merely a technicality which may be solved by suitable modifications of the bracket operations. For example, we can employ new bracket operations defined by $\{P_1, P_2\}' = -\{P_1, P_2\}$, $[\mathrm{P}_1, \mathrm{P}_2]' = [\mathrm{P}_1, \mathrm{P}_2]/(-\mathrm{i}\hbar)$. From now on we shall adopt these modified brackets.

as $V_{(\mu)}$, $\mu = 1, 2, \ldots, \frac{1}{2}N(N+1)$, by

$$V_{(i)} = L_i, \qquad V_{(r)} = L_{mn},$$

where $m < n$ and $r = Nm - \frac{1}{2}m(m+1) + n$. This is equivalent to the ordering of L_i, L_{mn} as

$$L_1, \ldots, L_N, L_{12}, \ldots, L_{1N}, L_{23}, \ldots, L_{2N}, \ldots, L_{(N-1)N}.$$

Notice that there is a one–one correspondence between r and m, n. Rewriting the Lie bracket relations in the form

$$[V_{(\mu)}, V_{(\nu)}] = C^{\lambda}_{\mu\nu} V_{(\lambda)}, \qquad \mu, \nu, \lambda = 1, 2, \ldots, \tfrac{1}{2}N(N+1)$$

$C^{\lambda}_{\mu\nu}$ are then the structure constants. There are only a few types of $C^{\lambda}_{\mu\nu}$ which do not vanish. They are

$$C^{r}_{ij} \equiv C^{(mn)}_{ij} = -K\delta^{mn}_{ij} + K\delta^{mn}_{ji}, \qquad C^{j}_{ir} \equiv C^{j}_{i(mn)} = \delta^{ij}_{mn} - \delta^{ji}_{mn},$$

$$C^{t}_{rs} \equiv C^{(ij)}_{(mn)(kl)} = \delta^{nij}_{kml} - \delta^{nij}_{lmk} + \delta^{nij}_{lkm} - \delta^{mij}_{knl} + \delta^{mij}_{kln} - \delta^{mij}_{lkn},$$

where

$$\delta^{mn}_{ij} = \delta^{m}_{i}\delta^{n}_{j}, \qquad \delta^{nij}_{kml} = \delta^{n}_{k}\delta^{i}_{m}\delta^{j}_{l}.$$

Apart from those derivable from the above non-vanishing constants the rest structure constants are all zero. Now let

$$G_{\mu\nu} = C^{\alpha}_{\mu\beta} C^{\beta}_{\nu\alpha}$$

then

$$G_{\mu\nu} = -2(N-1) \begin{pmatrix} \overset{\leftarrow N \rightarrow}{K} & 0 & & & & \\ 0 & \ddots & & & \bigcirc & \\ & & K & & & \\ & & & 1 & & \\ & \bigcirc & & & \ddots & 0 \\ & & & & 0 & 1 \end{pmatrix} \updownarrow \tfrac{1}{2}N(N+1).$$

Since $\det|G_{\mu\nu}| \neq 0$ (for the case $K \neq 0$) we can define $G^{\mu\nu}$ by

$$G^{\mu\alpha} G_{\alpha\nu} = \delta^{\mu}_{\nu}.$$

The result is

$$G^{\mu\nu} = 1/G_{\mu\nu} \quad \text{if} \quad \mu = \nu, \qquad G^{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu.$$

Hence a Casimir operator C_V is

$$C_V = G^{\mu\nu} V_{(\mu)} V_{(\nu)} = -[2(N-1)]^{-1} \Big(K^{-1} \sum_i L_i^2 + \sum_{i<j} L_{ij}^2 \Big).$$

We have $[V_{(\mu)}, C_V] = 0$ for every $V_{(\mu)}$. For a space of positive constant curvature C_V is the total angular momentum square apart from a multiplicative constant (Popov and Perelomov 1968).

An identical analysis on the algebras $\{P\}_k$ and $\{\mathrm{P}\}_k$ shows that

$$C = G^{\mu\nu} P_{(\mu)} P_{(\nu)}, \qquad \mathrm{C} = G^{\mu\nu} \mathrm{P}_{(\mu)} \mathrm{P}_{(\nu)}$$

are Casimir operators of $\{P\}_k$ and $\{\mathrm{P}\}_k$ respectively. Here $P_{(\mu)}$, $\mathrm{P}_{(\mu)}$ are respectively the classical and quantised momenta corresponding to $V_{(\mu)}$. In accordance with our scheme in I we conclude that the classical Hamiltonian H and its quantum counterpart H are simply C and C respectively apart from a multiplicative constant. To be precise we have

$$H = \tfrac{1}{2}m^{-1}g^{ij}p_ip_j = m^{-1}K(1-N)G^{\mu\nu}P_{(\mu)}P_{(\nu)},$$

$$\mathrm{H} = -\hbar^2/2m\ \nabla^2 = m^{-1}K(1-N)G^{\mu\nu}\mathrm{P}_{(\mu)}\mathrm{P}_{(\nu)}.$$

In terms of Casimir operators

$$H = m^{-1}K(1-N)C = \frac{R}{mN}C, \qquad \mathrm{H} = m^{-1}K(1-N)\mathrm{C} = \frac{R}{mN}\mathrm{C},$$

where $R = KN(1-N)$ is the curvature invariant (Synge and Schild 1966). The striking similarity between H and H is most gratifying.

4. Conclusion

The idea of Lie algebra is seen to play a significant role in quantisation especially in the construction of the Hamiltonian when the configuration space possesses geometry symmetry. But this should not blind us to the other conclusion that Lie algebra as described is not a basis for a general formulation of the problem of quantisation.

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